

Polygonal, Pyramidal and Hypersolid numbers

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1 Decomposition of polygonal numbers

In this section, we will formulate and prove a proposition that will enable us to obtain the integers sequences associated with each polygonal number.

Proposition

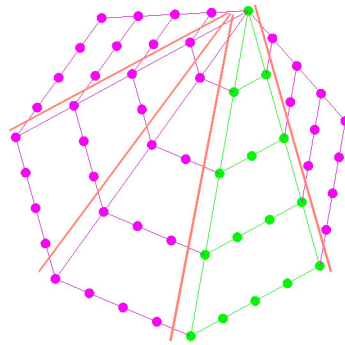
Each k -gonal number $p(k, n)$ can be decomposed into the sum of a triangular number $T(n)$ plus $k - 3$ triangular numbers $T(n - 1)$:

$$p(k, n) = T(n) + (k - 3)T(n - 1) \quad (1)$$

Inversely, each k -gonal number is given by the sum of a triangular number $T(n)$ plus $k - 3$ triangular numbers $T(n - 1)$.

Proof

Consider a regular polygon of k sides, that we associate to the figurate k -gonal(n) number. From a vertex of that polygon, track the joining line with other vertices, as in the heptagon example ¹ shown in the next figure:



Are thus obtained $k - 2$ triangular areas. If to any of these areas we associate the triangular number $T(n)$, each of the other $k - 3$ remaining areas are associated with the triangular number $T(n - 1)$, as can be seen by counting in the figure. Summing all triangles, we obtain the (1).

¹ This is an inductive process, so the reasoning is true for any number of sides k .

Note that polygonal numbers are produced by summing arithmetical progressions.

Substituting into (1) the triangular number formula: $T(n) = n(n+1)/2$, one obtains:

$$p(k, n) = n(n+1)/2 + (k-3)(n-1)((n-1)+1)/2$$

that factored becomes:

$$p(k, n) = n[(k-2)n - (k-4)]/2 \quad (2)$$

This is the same formula that appears on Wolfram MathWorld, page polygonal number, pos. (5), as it should be.

Formula (2) allows us to obtain all k -gonal numbers. We used it to get the following integer sequences, not yet present ² in the OEIS database:

$$25\text{-gonal number: } a(n) = n(23n - 21)/2$$

$$26\text{-gonal number: } a(n) = n(12n - 11)$$

$$27\text{-gonal number: } a(n) = n(25n - 23)/2$$

$$28\text{-gonal number: } a(n) = n(13n - 12)$$

$$29\text{-gonal number: } a(n) = n(27n - 25)/2$$

$$30\text{-gonal number: } a(n) = n(14n - 13)$$

Formula (1) allows us to construct the following *table of relations* between polygonal numbers:

Polygonal numbers	
Triangular (Tn)	
Square	= Triangular + T(n-1)
Pentagonal	= Square + T(n-1) = Triangular + 2T(n-1)
Hexagonal	= Pentagonal + T(n-1) = Square + 2T(n-1) = Triangular + 3T(n-1)
Heptagonal	= Hexagonal + T(n-1) = Pentagonal + 2T(n-1) = Square + 3T(n-1)
Octagonal	= Heptagonal + T(n-1) = Hexagonal + 2T(n-1) = Pentagonal + 3T(n-1)
Enneagonal	= Octagonal + T(n-1) = Heptagonal + 2T(n-1) = Hexagonal + 3T(n-1)
Decagonal	= Enneagonal + T(n-1) = Octagonal + 2T(n-1) = Heptagonal + 3T(n-1) = etc.
Hendecagonal	= Decagonal + T(n-1) = Enneagonal + 2T(n-1) = Octagonal + 3T(n-1)
Dodecagonal	= Hendecagonal + T(n-1) = Decagonal + 2T(n-1) = Enneagonal + 3T(n-1)
etc.	etc.

The table continue indefinitely in both directions. From it you can also derive, by performing substitutions, many other relationships, such as:

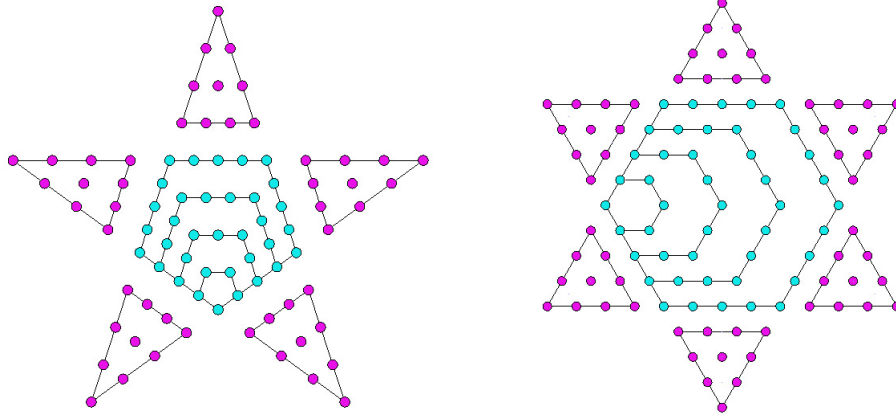
$$\text{Dod}(n) = \text{Enn}(n) - \text{Hep}(n) + \text{Hex}(n) + \text{Pen}(n) - \text{Tri}(n) + 2T(n-1)$$

² The 28-gonal number sequence A161935 is in the database, but with different name and meaning.

2 Star k-gonal numbers

We will show in this section how you can create, using polygonal numbers, a particular set of figurate numbers having star shape.

We will build these figurate numbers by placing on the sides of a k -gonal number, k triangular numbers, as in the cases $k = 5$ and $k = 6$ shown in the following figure:



As can be seen, the triangular numbers to be placed on the sides of polygons they must be always of the $n - 1$ order.

In our research, we just build figurate numbers using polygonal numbers from the 5-gonal to the 12-gonal, thus obtaining integer sequences, all stored in the OEIS database, but with different name and meaning.

We then added, in the comments section of the found sequences, the following annotations:

$$A001107(n) = A000326(n) + 5 \cdot A000217(n - 1)$$

$$A051624(n) = A000384(n) + 6 \cdot A000217(n - 1)$$

$$A051866(n) = A000566(n) + 7 \cdot A000217(n - 1)$$

$$A051868(n) = A000567(n) + 8 \cdot A000217(n - 1)$$

$$A051870(n) = A001106(n) + 9 \cdot A000217(n - 1)$$

$$A051872(n) = A001107(n) + 10 \cdot A000217(n - 1)$$

$$A051874(n) = A051682(n) + 11 \cdot A000217(n - 1)$$

$$A051876(n) = A051624(n) + 12 \cdot A000217(n - 1)$$

Observations

From what we saw in the above sections, it is apparent that:

1 - any k -gonal number is obtained from the previous $(k - 1)$ -gonal number by adding to it a triangular number $T(n - 1)$;

2 - by adding m times the triangular number $T(n - 1)$ to a k -gonal number, you get the $(k + m)$ -gonal number;

3 - by adding k times the triangular number $T(n - 1)$ to a k -gonal number, you get the $2k$ -gonal number, which is also the star k -gonal number³.

3 Pyramidal Numbers

Pyramidal numbers are obtained by piling successive polygonal numbers one upon the other. We will formulate and prove a proposition that will enable us to obtain integer sequences associated to each k -gonal pyramidal number.

Proposition

Each k -gonal pyramidal number $P(k, n)$ can be decomposed into the sum of a tetrahedral number $Te(n)$ plus $k - 3$ tetrahedral numbers $Te(n - 1)$:

$$P(k, n) = Te(n) + (k - 3)Te(n - 1) \quad (3)$$

Inversely, each k -gonal pyramidal number is given by the sum of a tetrahedral number $Te(n)$ plus $k - 3$ tetrahedral numbers $Te(n - 1)$.

Proof

The proof is carried out in a completely analogous way to that made in Section 1 of this article. The roles of $T(n)$ and $T(n - 1)$ are carried out here by $Te(n)$ and $Te(n - 1)$ respectively. Vertices become edges, sides become faces, triangles become pyramids. A graphic representation is difficult to achieve, but easy to imagine: see to a succession of layers stacked to form a pyramid, each of which represents the polygonal number $p(k, n)$ ($n = 1, 2, \dots, n$). So then, pyramidal numbers will be given by partial sums of polygonal numbers:

$$P(k, n) = \sum_{q=1}^n p(k, q) = \sum_{q=1}^n Te(q) + (k - 3) \sum_{q=1}^n Te(q - 1)$$

ie, the proposition to be proven.

Substituting in (3) the tetrahedral number formula: $Te(n) = n(n + 1)(n + 2)/6$, one obtains:

³ This follows from the topological identity of figures.

$$P(k, n) = n(n+1)(n+2)/6 + (k-3)n(n-1)(n+1)/6$$

that factored becomes:

$$P(k, n) = n(n+1)[(k-2)n + (5-k)]/6 \quad (4)$$

This is the same formula that appears on Wolfram MathWorld, page Pyramidal Number, pos. (1), as it should be.

Formula (4) allows us to obtain all k -gonal pyramidal numbers. We used it to get the following integer sequences, not yet present in the OEIS database :

$$25\text{-gonal pyramidal number: } a(n) = n(n+1)(23n-20)/6$$

$$26\text{-gonal pyramidal number: } a(n) = n(n+1)(8n-7)/2$$

$$27\text{-gonal pyramidal number: } a(n) = n(n+1)(25n-22)/6$$

$$28\text{-gonal pyramidal number: } a(n) = n(n+1)(26n-23)/6$$

$$29\text{-gonal pyramidal number: } a(n) = n(n+1)(9n-8)/2$$

$$30\text{-gonal pyramidal number: } a(n) = n(n+1)(28n-25)/6$$

Even here, as was done at the end of Section 1, it would be possible to organize a table to derive complex relationships between k -gonal pyramidal numbers.

4 Four-dimensional solid numbers

In the preceding section 3 we saw how, by performing partial sums of plan polygonal numbers, you get the corresponding solid pyramidal numbers. This procedure can be extended to generate solid numbers in higher dimensions. Thus, by piling successive pyramidal numbers, one obtains four-dimensional solid numbers, whose base is the n -th pyramidal number.

Consider the hypertetrahedral number obtained by summing tetrahedral numbers:

$$Z(n) = \sum_{q=1}^n Te(n) = n(n+1)(n+2)(n+3)/24 \quad (5)$$

We will use this figurate number to reformulate propositions (1) and (3) in $4D$ space. Let $F(k, n)$ denote the n -th four-dimensional solid number, then one can say:

Each $4D$ solid number $F(k, n)$ can be decomposed into the sum of a hypertetrahedral number $Z(n)$ plus $k-3$ hypertetrahedral numbers $Z(n-1)$:

$$F(k, n) = Z(n) + (k - 3)Z(n - 1) \quad (6)$$

Inversely, each 4D solid number is given by the sum of an hypertetrahedral number $Z(n)$ plus $k - 3$ hypertetrahedral numbers $Z(n - 1)$.

In fact, from (3) one obtains by summing:

$$F(k, n) = \sum_{q=1}^n Te(n) + (K - 3) \sum_{q=1}^n Te(n - 1)$$

ie, the proposition to be proven.

Substituting into (6) the hypertetrahedral number formula (5), and factoring, you get the general formula:

$$F(k, n) = n(n + 1)(n + 2)[(k - 2)n - (k - 6)]/24 \quad (7)$$

This formula allows to get all four-dimensional solid numbers.

5 Multidimensional solid numbers

In summary, we have so far achieved the following general formulas:

$$2D \text{ space: } p(k, n) = n[(k - 2)n - (k - 4)]/2$$

$$3D \text{ space: } P(k, n) = n(n + 1)[(k - 2)n - (k - 5)]/6$$

$$4D \text{ space: } F(k, n) = n(n + 1)(n + 2)[(k - 2)n - (k - 6)]/24$$

that we can generalize to subsequent j -dimensional spaces:

$$S(j, k, n) = \frac{n(n + 1)(n + 2) \dots (n + j - 2)}{2 \times 3 \times 4 \dots \times j} [(k - 2)n - (k - (j + 2))]$$

where $S(j, k, n)$ denote the n -th hypersolid number in the j -dimensional space.

6 Diagonal sequences

Consider the general formula derived in Section 1:

$$p(k, n) = n[(k - 2)n - (k - 4)]/2 \quad (8)$$

As n and k vary, this formula generates the following table of polygonal numbers:

n	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=12$	$k=13$
1	1	1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11	12	13
3	6	9	12	15	18	21	24	27	30	33	36
4	10	16	22	28	34	40	46	52	58	64	70
5	15	25	35	45	55	65	75	85	95	105	115
6	21	36	51	66	81	96	111	126	141	156	171
7	28	49	70	91	112	133	154	175	196	217	238
8	36	64	92	120	148	176	204	232	260	288	316
9	45	81	117	153	189	225	261	297	333	369	405
10	55	100	145	190	235	280	325	370	415	460	505
11	66	121	176	231	286	341	396	451	506	561	616

Columns contain sequences of k -gonal numbers, starting from triangular numbers ($k = 3$). We look the yellow sequence in the main diagonal of the table. The formula of this sequence is derived from (8) by substituting $k = n + 2$ and factoring. You get:

$$d_2(n) = n(n^2 - n + 2)/2$$

With $k = n + 3$, you get the adjacent diagonal sequence, and so on.

Similarly, from other general formulas listed in Section 5, are obtained other hypersolid numbers tables and other formulas for related diagonal sequences. Even these formulas are generalizable by recurrence. In fact, we have:

$$d_3(n) = n(n + 1)(n^2 - n + 3)/6$$

$$d_4(n) = n(n + 1)(n + 2)(n^2 - n + 4)/24$$

In general:

$$d_j(n) = \frac{n(n + 1)(n + 2) \dots (n + j - 2)}{2 \times 3 \times 4 \dots \times j} (n^2 - n + j)$$

7 Diagonal sums

Let us examine again the above table. Another interesting sequence is that obtained by summing polygonal numbers diagonally, starting from the top left. You get: 1, 4, 11, 25, 50,..., whose formula is:

$$D_2(n) = n(1 + n)(10 + n + n^2)/24$$

This formula is obtained from "Mathematica" with the following command:

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Factor[FindSequenceFunction[{1, 4, 11, 25, 50, ...}, n]]
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Inserting sequences of successive j -rank spaces, one obtains:

$$D_3(n) = n(1+n)(2+n)(n^2+2n+17)/120$$

$$D_4(n) = n(1+n)(2+n)(3+n)(n^2+3n+26)/720$$

In general:

$$D_j(n) = \frac{n(n+1)(n+2)\dots(n+j-1)}{2 \times 3 \times 4 \dots \times (j+2)} [n^2 + (j-1)n + ((j+1)^2 + 1)]$$

8 Conclusions

Concluding this brief structural analysis, we can say that, using unit bricks $T(n-1)$, we can construct, above the basic number $T(n)$, all the possible k -gonal and star k -gonal numbers. In $3D$ space, using unit bricks $Te(n-1)$, we can construct, above the basic number $Te(n)$, all the possible k -gonal pyramidal numbers. Likewise, in j -dimensional spaces, using unit bricks corresponding to $k=3$, we can construct all the possible hypersolid numbers.